We are now able to pass from any particular redundant generating function to an equivalent generating function which involves $n-1$ undetermined quantities. Assuming these quantities at pleasure, we obtain a number of different algebraic products, each of which may have its own meaning in arithmetic, and thus the number of arithmetical correspondences obtainable is subject to no finite limit.

This portion of the theory is given at length in the paper, with illustrative examples.

Incidentally interesting results are obtained in the fields of special and general determinant theory. The special determinant, which presents itself for examination, provisionally termed "inversely symmetric," is such that the constituents symmetrically placed in respect to the principal axis have, each pair, a product unity, whilst the constituents on the principal axis itself are all of them equal to unity. The determinant possesses many elegant properties which are of importance to the principal investigation of the paper. The theorems concerning the general determinant are connected entirely with the co-axial minors.

I find that the general determinant of even order, greater than two, is expressible in precisely two ways as an irrational function of its co-axial minors, whilst no determinant of uneven order is so expressible at all.

Of order superior to 3, it is not possible to assume arbitrary values for the determinant itself and all of its co-axial minors. In fact of order n the values assumed must satisfy

$$2^n - n^2 + n - 2$$

conditions, but, these conditions being satisfied, the determinant can be constructed so as to involve $n-1$ undetermined quantities.

IV. "On the Whirling and Vibration of Shafts." By STANLEY DUNKERLEY, M.Sc., Berkeley Fellow of the Owens College, Manchester. Communicated by OSBORNE REYNOLDS, F.R.S. Received July 13, 1893.

(Abstract.)

It is well known that every shaft, however nearly balanced, when driven at a particular speed bends, and, unless the amount of deflection be limited, might even break, although at higher speeds the shaft again runs true. The particular or "critical" speed depends on the manner in which the shaft is supported, its size and modulus of elasticity, and the size, weights and positions, of any pulleys it carries.

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The theory for the case of an unloaded shaft first received attention at the hands of Professor Rankine,* who obtained numerical formulæ for the cases of a shaft resting freely on a bearing at each end, and for an overhanging shaft fixed in direction at one end.

The theory has been further extended to the case of a shaft loaded with pulleys by Professor Reynolds; and the object of this investigation is to apply that theory and so obtain formulæ—and by experiment to verify them—giving the critical speed in terms of the diameter of the shaft, weights of pulleys, &c., in particular cases applicable to the different conditions under which a shaft works.

In many cases, as might naturally be expected, the "period of whirl" of the shaft is merely its natural period of vibration. The two periods are coincident in the case of an unloaded shaft (however supported), and for a loaded shaft on which the pulleys are placed in such positions that they rotate, when the shaft is whirling, in planes perpendicular to the original alignment of the shaft. With pulleys placed in any other positions, when the shaft is whirling, there is a righting moment tending to straighten the shaft which does not exist when it merely vibrates under the dead weight of the pulleys. Hence, in an unloaded shaft, the period of whirl coincides with the natural period of vibration; but, generally, in a loaded shaft, the period of whirl is less than the natural period of vibration to an extent depending on the size and positions of the pulleys.

If, therefore, the period of disturbance (that is, the period of one revolution) be decreased, the shaft runs true until that period approximates to the natural period of vibration of the shaft under the given conditions. If the shaft now receive any displacement, however slight, a violent agitation is set up, which will be most marked when the period of disturbance and the whirling period coincide. As the period of disturbance is further decreased, the agitation becomes less and, at a period of disturbance slightly less than the whirling period of the shaft, the shaft again runs true.

As in the vibration of rods, so in the whirling of shafts, there are a series of periods at which the shaft whirls.

Investigation shows that the formulæ obtained by considering the combined effects of the shaft and only one pulley, or the combined effects of two pulleys neglecting the effect of the shaft, are too complicated—even in the simplest cases—for actual use. The only alternative method is to consider the effects of the shaft and each of the pulleys (whatever be their number, position, and size) separately, and so obtain the whirling speed for each on the assumption that all the others are neglected. By means of an empirical formula, the whirling speed taking shaft and pulleys into account may be calculated from the separately calculated speeds of whirl.

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* Rankine's 'Machinery and Millwork,' p. 549.

Unloaded Shaft.—In the case of an unloaded shaft, the whirling speed is given by an equation of the form

$$(w\omega^2/g\mathrm{EI})^{\frac{1}{4}}l = a \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

in which

w = weight of shaft in lbs. per foot run,
g = gravity,
E = Young's Modulus (in lbs. per sq. ft.),
I = geometrical moment of inertia of cross section of shaft about a diameter,
ω = angular velocity of shaft when whirling,
l = length in feet of one of the spans, and
a = some numerical coefficient depending on the manner in which the shaft is supported.

By substituting the proper value for the coefficients in Equation (1), we obtain the equations

$$\mathrm{N} = b\ d/l^2, \text{ for a solid shaft} \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

and

$$= b\sqrt{(d_1^2 + d_2^2)}/l^2, \text{ for a hollow shaft} \ \ldots\ldots \quad (3).$$

In these equations

N = number of revolutions shaft makes, per minute, when whirling,
d = diameter of the solid shaft in inches,
d_1, d_2 = the external and internal diameters of the hollow shaft, both in inches, and
b = some numerical coefficient depending on a, and also upon the constants in Equation (1).

For wrought iron or mild steel, $b = 3331a^2$ and for brass, $2222a^2$.

It should be noticed that each span of a continuous shaft supported on bearings placed at equal distances apart whirls independently of the rest.

Loaded Shafts.—The cases for a loaded shaft—as also those for an unloaded one—which have been considered in the investigation, embrace all those which are likely to arise in practice. With a continuous shaft supported on three or more bearings and loaded with a pulley on one of the spans, general equations have been obtained for unequal spans, but the complete investigation has only been extended to the case of equal spans—a relation between the spans which almost invariably exists in actual shafting. The results of the investigation show that however the shaft be supported, the whirling speed due to a single pulley on a light shaft is given by an equation of the form

$$\omega = \theta\sqrt{(g\mathrm{EI}/\mathrm{W}c^3)} \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

in which ω, g, E, I have the same meanings as in Equation (1), and

W = weight of pulley in lbs.;
c = distance of pulley from nearest bearing in feet; and
θ = some numerical coefficient depending not only upon the manner in which the shaft is supported, but also upon the position of the pulley on the span which carries it and the size of the pulley.

For any particular mode of support of the shaft the coefficient θ is some function of the ratios c/l and c/k, where l = length of span which carries the pulley, and $k = \sqrt{g\,(A-B)/W}$, where A and B are the moments of inertia of the pulley about the axis of the shaft and a diameter of the pulley through its centre of gravity respectively, both being expressed in gravitation units. Assuming, therefore, certain values for c/l and c/k, tables have been drawn up giving the corresponding values of θ. For each value of c/l there are two limiting values of c/k, viz., *infinity* and *zero*—the corresponding values of k being *zero* and *infinity*. When $k = 0$ the pulley may be considered simply as a dead weight, so that the "inferior period of whirl," as it has been termed, is the natural period of vibration of the light shaft under the given conditions. The "superior period of whirl," that is to say, the whirling period when $k = \infty^y$, is the inferior period multiplied by some function of c/l, and assuming the shaft to whirl at a speed corresponding to the superior or inferior limit, it would do so in such a manner that the pulley still rotated in a plane perpendicular to the original alignment of the shaft. These limiting values of the speed, below and above which whirling is impossible, have been calculated in each case.

Investigation shows that in a continuous light shaft supported on bearings placed at equal distances apart, the increase in the whirling speed due to those spans which are not immediately adjacent to the loaded one, on either side, cannot exceed by above 2 or 3 per cent. the whirling speed when only the loaded span and the spans immediately adjacent to it are taken into account. In other words, the stiffening effects of only those spans immediately adjacent to the loaded one, on either side, need be taken into account in calculating the whirling speed for the shaft.

By substituting the proper values for the constants in Equation (4) we obtain the equations

$$N = \phi\, d^2/\sqrt{Wc^3}, \text{ for a solid shaft} \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

and

$$= \phi\, \sqrt{(d_1^4 - d_2^4)}/\sqrt{Wc^3}, \text{ for a hollow shaft} \ldots\ldots \quad (6),$$

where W, d, d_1, d_2 have the same values as before, and ϕ is some numerical coefficient depending on the value of θ and of the constants in Equation (4).

The value of ϕ for wrought-iron or mild steel is $5389\,\theta$, and for brass $3772\,\theta$.

Having thus obtained formulæ giving the whirling speed due to each cause, on the assumption that all the others are neglected, it only remains to find an empirical formula giving the resulting whirling speed, when all the disturbing elements are taken into account, in terms of the separately calculated whirling speeds due to the several causes. Since the whirling speed in every case varies inversely as the square root of the weight of the pulley (*see* Equation 4), the formula for calculating the resulting speed was taken to be of the form

$$N_1N_2/\sqrt{(N_1^2+N_2^2)} \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

for two disturbing elements whose whirling speeds taken separately are N_1, N_2; or of

$$N_1N_2N_3/\sqrt{(N_2^2N_3^2+N_3^2N_1^2+N_1^2N_2^2)} \quad\ldots\ldots\ldots\ldots \quad (8)$$

for three disturbing elements whose whirling speeds, taken separately, are N_1, N_2, N_3. The formula may be extended to any number of disturbing elements. It is not strictly accurate, for in addition to the whirling speed varying inversely as the square root of the weight of the pulley, it also varies as some function (θ) of the distance of the pulley from the nearest bearing and of the size of the pulley. The experiments, however, justify to a remarkable degree the assumptions that have been made in calculating the resulting whirling speed.

In calculating the speed at which a continuous shaft of given diameter, supported on bearings placed at equal distances apart, and loaded with pulleys on any or all of the spans, will whirl, the method to adopt is to, first, find the span which will have the biggest whirl (that is to say the span which carries the heaviest and most advantageously-placed pulleys as regards whirling), and then to consider this span and the spans immediately adjacent on either side. The whirling speeds for the shaft and each of the pulleys on the three spans have then to be calculated according to the rules laid down in each case. The whirling speed for any pulleys on the two side spans will, of course, be different according as that side span is an end or an intermediate span in the line of shafting. Having found the whirling speed due to each cause, the resulting whirling speed is found from an equation of the same form as Equation (7) or (8)—the exact equation depending on the number of disturbing elements. The speed thus obtained will be slightly less than the actual whirling speed. A nearer approximation to the actual speed might be obtained by considering only those pulleys which lie near the centres, or between the centres of the side spans and the bearings of the middle span, neglecting the effect of those pulleys which lie beyond the centres of the side spans. In doing so, however, the experiments show that there is a danger of the calculated speed *exceeding* the actual, whilst by taking *all* the pulleys on the two side spans into account the calculated speed will be slightly *less* than the actual speed.

If the spans of a continuous shaft, supported on bearings placed at equal distances apart, are all loaded in the same manner, each whirls independently of the rest, and the problem reduces to that of a loaded shaft supported on bearings at the ends.

The *experimental apparatus* by which the calculated results have been, for the most part, verified is shown in a figure. The experimental shaft was 2 ft. 8 in. long and 0·2488 in. diameter. The motion was transmitted from the headstock spindle to the experimental shaft by a fine piece of steel wire (about $1\frac{1}{2}$ in. long and 21 B.W.G. diam.), so that the shaft was subjected to very little constraint at the end. The experimental pulleys were models of actual pulleys—being designed for both weight and inertia. The headstock spindle was driven from a turbine, the constancy of the speed being shown by the steadiness of a column of liquid forced by a centrifugal fan indicator up a glass tube. In taking the number of revolutions corresponding to any period of whirl an ordinary counter pushed into the end of the headstock spindle was used. In making any experiment three trials were made (each of three minutes' duration) and the mean of the results taken. Over 150 experiments have been made with this apparatus, and the observed results invariably approximate very closely to the calculated results. Experiments have also been made with actual cases of shafting, and it would appear that, following the method of solution sketched above, the calculated speed is about 3 or 4 per cent. *less* than the actual speed.

The experiments were carried out in the Whitworth Engineering Laboratory, the Owens College, Manchester.

V. "On Plane Cubics." By CHARLOTTE ANGAS SCOTT, D.Sc. (Lond.), Professor of Mathematics at Bryn Mawr College, Pennsylvania. Communicated by A. R. FORSYTH, Sc.D., F.R.S. Received September 9, 1893.

(Abstract.)

In this paper the first few sections are devoted to certain constructions for the cubic, its Hessian, and its Cayleyan. Assuming three collinear inflexions for the cubic, and the tangents at these points, *i.e.*, eight conditions, one more point determines the cubic, and, consequently, also the Hessian and Cayleyan. Taking